CHAPTER 1

# HILBERT SPACES

The aim of this chapter is to show the Minlos-Sazanov theorem and deduce a characterization of Gaussian measures on separable Hilbert spaces by its Fourier transform. By using the notion of the Hellinger integral we prove the Kakutani theorem on infinite product measures. As a consequence we obtain the Cameron-Martin theorem.

For Gaussian measures on Banach spaces and their relationship with parabolic equations with many infinitely variables we refer to [22] and [12] and the references therein.

## 1.1 BOREL MEASURES ON HILBERT SPACES

Let $H$ be a real separable Hilbert space, $\mathcal{B}(H)$ the Borel $\sigma$-algebra on $H$. Then $\mathcal{B}(H)$ is a separable $\sigma$-algebra. A measure on the measurable space $(H, \mathcal{B}(H))$ is called a **Borel measure** on $H$. Here we only investigate finite Borel measures.

**Definition 1.1.1** *Let $\mu$ be a finite Borel measure on $H$. The* **Fourier transform** *of $\mu$ is defined by*

$$\widehat{\mu}(x) := \int_H e^{i<x,y>} \mu(dy), \qquad x \in H.$$

Clearly $\widehat{\mu}$ possesses the following properties.

**Proposition 1.1.2** *The Fourier transform of a finite Borel measure satisfies the following properties*

*(1)* $\widehat{\mu}(0) = \mu(H)$.

*(2)* $\widehat{\mu}$ *is continuous on* $H$.

*(3)* $\widehat{\mu}$ *is positive definite in the sense that*

$$\sum_{l,k=1}^{n} \widehat{\mu}(x_l - x_k)\alpha_l\overline{\alpha_k} \geq 0. \tag{1.1}$$

*for any* $n \geq 1$, $x_1, x_2, \cdots, x_n \in H$, *and* $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{C}$.

**Proof:** We have only to prove the third assertion. For $n \geq 1$, $x_1, x_2, \ldots, x_n \in H$, and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$ we have

$$\begin{aligned}
\sum_{l,k=1}^{n} \widehat{\mu}(x_l - x_k)\alpha_l\overline{\alpha_k} &= \sum_{l,k=1}^{n} \int_H e^{i<x_l,y>} e^{-i<x_k,y>} \alpha_l\overline{\alpha_k}\mu(dy) \\
&= \sum_{l,k=1}^{n} \int_H \left(e^{i<x_l,y>}\alpha_l\right) \overline{\left(e^{i<x_k,y>}\alpha_k\right)}\mu(dy) \\
&= \langle \sum_{l=1}^{n} e^{i<x_l,\cdot>}\alpha_l, \sum_{k=1}^{n} e^{i<x_k,\cdot>}\alpha_k \rangle_{L^2(H,\mu)} \\
&= \int_H \left| \sum_{k=1}^{n} e^{i<x_k,y>}\alpha_k \right|^2 \mu(dy) \geq 0.
\end{aligned}$$

Here $L^2(H, \mu)$ denotes the space of all measurable functions $f : H \to \mathbb{R}$ satisfying

$$\int_H |f(x)|^2 \mu(dx) < \infty.$$

□

A natural question arises. Is any positive definite continuous functional on $H$ the Fourier transform of some finite Borel measure?
The answer is affirmative if $\dim H < \infty$. This is exactly the classical Bochner theorem (see Theorem A.1.3). But in the infinite dimensional case the answer is negative. Take, for example,

$$\phi(x) := \exp\left(-\frac{1}{2}|x|^2\right), \qquad x \in H.$$

Then it is easy to see that $\phi$ is a positive definite functional on $H$. But $\phi$ is not the Fourier transform of any finite Borel measure on $H$ as we will see later (see Proposition 1.2.11).
To this end let us prove some auxiliary results.

**Lemma 1.1.3** *Let* $\phi$ *be a positive definite functional on* $H$*. Then, for any* $x$, $y \in H$,

(1) $|\phi(x)| \leq \phi(0)$, $\overline{\phi(x)} = \phi(-x)$.

(2) $|\phi(x) - \phi(y)| \leq 2\sqrt{\phi(0)}\sqrt{\phi(0) - \phi(x-y)}$.

(3) $|\phi(0) - \phi(x)| \leq \sqrt{2\phi(0)(\phi(0) - \Re(\phi)(x))}$.

**Proof:** For $x, y \in H$, set

$$A := \begin{pmatrix} \phi(0) & \phi(x) \\ \phi(-x) & \phi(0) \end{pmatrix}$$

$$B := \begin{pmatrix} \phi(0) & \phi(x) & \phi(y) \\ \phi(-x) & \phi(0) & \phi(y-x) \\ \phi(-y) & \phi(x-y) & \phi(0) \end{pmatrix}$$

Since $\phi$ is positive definite, one can see that both $A$ and $B$ are positive definite matrices. In particular $\overline{A}^t = A$. Hence, $\overline{\phi(x)} = \phi(-x)$ for all $x \in H$. From $\det(A) \geq 0$, it follows that $|\phi(x)| \leq \phi(0)$.
On the other hand, we have

$$\begin{aligned} \det B &= \phi(0)^3 - \phi(0)|\phi(x-y)|^2 - \phi(x)[\phi(0)\overline{\phi(x)} - \overline{\phi(x-y)\phi(y)}] + \\ &\quad \phi(y)[\overline{\phi(x)}\phi(x-y) - \phi(0)\overline{\phi(y)}] \\ &= \phi(0)^3 - \phi(0)|\phi(x-y)|^2 - \phi(0)|\phi(x) - \phi(y)|^2 + \\ &\quad 2\Re[\phi(y)\overline{\phi(x)}(\phi(x-y) - \phi(0))]. \end{aligned}$$

Using the inequality $a^3 - ab^2 \leq 2a^2|a-b|$ for $|b| < a$, we find

$$\phi(0)^3 - \phi(0)|\phi(x-y)|^2 \leq 2\phi(0)^2|\phi(0) - \phi(x-y)|.$$

Therefore,

$$0 \leq \det B \leq 4\phi(0)^2|\phi(0) - \phi(x-y)| - \phi(0)|\phi(x) - \phi(y)|^2$$

This proves (2).
Finally (3) follows from

$$\begin{aligned} |\phi(0) - \phi(x)|^2 &= (\phi(0) - \phi(x))\left(\phi(0) - \overline{\phi(x)}\right) \\ &= \phi(0)^2 - 2\Re(\phi(0)\phi(x)) + |\phi(x)|^2 \\ &\leq 2\phi(0)^2 - 2\phi(0)\Re(\phi)(x). \end{aligned}$$

□

The following lemma will be useful for the proof of the Minlos-Sazanov theorem.

**Lemma 1.1.4** *Let $\mu$ be a finite Borel measure on $H$. Then the following assertions are equivalent.*

(i) $\int_H |x|^2 \mu(dx) \ < \infty$.

(ii) *There exists a positive, symmetric, trace class operator* $Q$ *such that for* $x, y \in H$

$$\langle Qx, y \rangle = \int_H \langle x, z \rangle \langle y, z \rangle \mu(dz). \tag{1.2}$$

*If (ii) holds, then* $\mathrm{Tr} Q = \int_H |x|^2 \mu(dx)$.

**Proof:** Suppose that (ii) holds. Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis of $H$. Then

$$\int_H |x|^2 \mu(dx) = \sum_{n=1}^{\infty} \int_H |\langle x, e_n \rangle|^2 \mu(dx) = \sum_{n=1}^{\infty} \langle Qe_n, e_n \rangle = \mathrm{Tr} Q < \infty. \tag{1.3}$$

Conversely, assume that (i) is satisfied. Thus,

$$\int_H |\langle x, z \rangle \langle y, z \rangle| \mu(dz) \ \le |x||y| \int_H |z|^2 \mu(dz).$$

By the Riesz representation theorem there exists $Q \ \in \mathcal{L}(H)$ such that (1.2) is satisfied. Obviously, $Q$ is positive and symmetric. Furthermore, by (1.3),

$$\mathrm{Tr} Q = \int_H |x|^2 \mu(dx) < \infty.$$

Hence $Q$ is of trace class. □

Let show now the Minlos-Sazanov theorem.

**Theorem 1.1.5** *Let* $\phi$ *be a positive definite functional on a separable real Hilbert space* $H$. *Then the following assertions are equivalent.*

(1) $\phi$ *is the Fourier transform of a finite Borel measure on* $H$.

(2) *For every* $\varepsilon > 0$ *there is a symmetric positive operator of trace class* $Q_\varepsilon$ *such that*

$$\langle Q_\varepsilon x, x \rangle \ < 1 \implies \Re(\phi(0) - \phi(x)) < \varepsilon.$$

(3) *There exists a positive symmetric operator of trace class* $Q$ *on* $H$ *such that* $\phi$ *is continuous (or, equivalently, continuous at* $x = 0$*) with respect to the semi-norm* $|\cdot|_Q$, *where*

$$|x|_Q := \sqrt{\langle Qx, x \rangle} = |Q^{1/2} x|, \quad x \in H.$$

**Proof:** (1) $\Longrightarrow$ (2): Let $\phi = \widehat{\mu}$. By applying the inequality

$$2(1 - \cos \vartheta) \le \vartheta^2, \qquad \forall\, \vartheta \in \mathbb{R},$$

we obtain, for any $\gamma > 0$,

$$\begin{aligned}\Re(\phi(0)-\phi(x)) &= \int_H (1-\cos\langle x,z\rangle)\ \mu(dz)\\ &\le \frac{1}{2}\int_{\{|z|\le\gamma\}} \langle x,z\rangle^2\ \mu(dz) + 2\mu\left(\{z : |z|>\gamma\}\right).\end{aligned}$$

Set $\mu_1(A) := \mu(A\cap\{|z|\le\gamma\})$ for $A\in\mathcal{B}(H)$, and apply Lemma 1.1.4 to $\mu_1$. Thus there is a positive symmetric operator of trace class $B_\gamma$ such that

$$\langle B_\gamma z_1, z_2\rangle = \int_{\{|z|\le\gamma\}} \langle z_1,z\rangle\langle z_2,z\rangle\ \mu(dz).$$

On the other hand, for every $\varepsilon>0$ there is $\gamma>0$ such that $\mu(\{z : |z| > \gamma\}) \le \frac{\varepsilon}{4}$. Put $Q_\varepsilon := \frac{1}{\varepsilon}B_\gamma$, then

$$\Re\left(\phi(0)-\phi(x)\right) \le \frac{\varepsilon}{2}\langle Q_\varepsilon x, x\rangle + \frac{\varepsilon}{2}.$$

(2) $\Longrightarrow$ (1): Assume that (2) holds. Then $\Re(\phi)(x)$ is continuous at $x=0$. So, by Lemma 1.1.3-(2), $\phi$ is continuous on $H$.
Now, take any orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of $H$ and for $n\ge1$ put

$$f_{i_1,\cdots,i_n}(\omega_1,\cdots,\omega_n) : \phi(\omega_1 e_1+\cdots+\omega_n e_n), \qquad \omega_j\in\mathbb{R},\ 1\le j\le n.$$

Then $f_{i_1,\cdots,i_n}$ is a positive definite function on $\mathbb{R}^n$. By the classical Bochner theorem (see Theorem A.1.3) there exists a finite Borel measure $\mu_{i_1,\cdots,i_n}$ on $\mathbb{R}^n$ such that

$$f_{i_1,\cdots,i_n} = \widehat{\mu}_{i_1,\cdots,i_n}.$$

The family $\{\mu_{i_1,\cdots,i_n}\}$ satisfies the *consistency conditions of Kolmogorov's extension theorem* for measures (cf. [30], p. 144). Hence there is a unique finite Borel measure $\gamma$ on $(\mathbb{R}^\infty,\mathcal{B}(\mathbb{R}^\infty))$ such that

$$\mu_{i_1,\cdots,i_n} = \gamma\circ(X_{i_1},\cdots,X_{i_n})^{-1},$$

where $\gamma\circ(X_{i_1},\cdots,X_{i_n})^{-1}$ is defined by

$$\gamma\circ(X_{i_1},\cdots,X_{i_n})^{-1}(A) = \gamma((X_{i_1},\cdots,X_{i_n})^{-1}(A)) \quad \text{for } A\in\mathcal{B}(H),$$

and $X_j(\omega)=\omega_j$, $\omega=(\omega_1,\cdots,\omega_n,\cdots)\in\mathbb{R}^\infty$, $j\in\mathbb{N}$.
Claim: $\sum_{k=1}^\infty X_k^2 < \infty$ $\gamma$-a.e..
Let $\mathbb{P}_n$ be the standard Gaussian measure on $\mathbb{R}^n$. Then

$$\int_{\mathbb{R}^n} e^{i(a_1y_1+\cdots+a_ny_n)}\mathbb{P}_n(dy) = \exp\left(-\frac{1}{2}\sum_{j=1}^n a_j^2\right).$$

By assumption, we know that for every $\varepsilon > 0$ there is a positive symmetric operator $Q_\varepsilon$ of trace class such that

$$\langle Q_\varepsilon x, x\rangle < 1 \Rightarrow \Re\left(\phi(0) - \phi(x)\right) < \varepsilon.$$

Hence, by Lemma 1.1.3-(1),

$$\phi(0) - \Re(\phi)(x) \leq \varepsilon + 2\phi(0)\langle Q_\varepsilon x, x\rangle \quad \text{for } x \in H.$$

By Fubini's theorem we obtain

$$\begin{aligned}
&\phi(0) - \int_{\mathbb{R}^\infty} \exp\left(-\frac{1}{2}\sum_{j=1}^n X_{k+j}^2\right)\gamma(d\omega)\\
= \quad & \phi(0) - \int_{\mathbb{R}^\infty} \gamma(d\omega)\int_{\mathbb{R}^n} \exp\left(i\sum_{j=1}^n y_j X_{k+j}\right)\mathbb{P}_n(dy)\\
= \quad & \phi(0) - \int_{\mathbb{R}^n} \mathbb{P}_n(dy)\int_{\mathbb{R}^\infty} \exp\left(i\sum_{j=1}^n y_j X_{k+j}\right)\gamma(d\omega)\\
= \quad & \phi(0) - \int_{\mathbb{R}^n} \mathbb{P}_n(dy)\phi\left(\sum_{j=1}^n y_j e_{k+j}\right)\\
= \quad & \int_{\mathbb{R}^n} [\phi(0) - \Re(\phi)(\sum_{j=1}^n y_j e_{k+j})]\mathbb{P}_n(dy)\\
\leq \quad & \varepsilon + 2\phi(0)\int_{\mathbb{R}^n} \langle Q_\varepsilon \sum_{j=1}^n y_j e_{k+j}, \sum_{l=1}^n y_l e_{k+l}\rangle \mathbb{P}_n(dy)\\
= \quad & \varepsilon + 2\phi(0)\sum_{l,j=1}^n \langle Q_\varepsilon e_{k+j}, e_{l+j}\rangle \int_{\mathbb{R}^n} y_j y_l \mathbb{P}_n(dy)\\
= \quad & \varepsilon + 2\phi(0)\sum_{j=1}^n \langle Q_\varepsilon e_{k+j}, e_{k+j}\rangle \underbrace{\int_{\mathbb{R}^n} y_j^2 \mathbb{P}_n(dy)}_{=1}\\
= \quad & \varepsilon + 2\phi(0)\sum_{j=1}^n \langle Q_\varepsilon e_{k+j}, e_{k+j}\rangle.
\end{aligned}$$

Hence,

$$\phi(0) - \int_{\mathbb{R}^\infty} \exp\left(-\frac{1}{2}\sum_{j=1}^n X_{k+j}^2\right)\gamma(d\omega) \leq \varepsilon + 2\phi(0)\sum_{j=k+1}^\infty \langle Q_\varepsilon e_j, e_j\rangle.$$

Now, let $k \longrightarrow \infty$, and $\varepsilon \longrightarrow 0$, so we get

$$\lim_{k \to +\infty} \int_{\mathbb{R}^\infty} \exp\left(-\frac{1}{2}\sum_{j=k+1}^{\infty} X_j^2\right) \gamma(d\omega) = \phi(0)\ (= \gamma(\mathbb{R}^\infty) \neq 0).$$

This means that the function $\exp(-\frac{1}{2}\sum_{j=k+1}^{\infty} X_j^2)$ converges in $L^1(\mathbb{R}^\infty, \gamma)$ to the constant function $1$. Thus there is a subsequence of

$$\exp(-\frac{1}{2}\sum_{j=k+1}^{\infty} X_j^2)$$

converging to $1$ $\gamma$–a.e., which implies that

$$\sum_{j=1}^{\infty} X_j^2 < \infty\ \gamma - \text{a.e.},$$

and the claim is proved.
Finally, let

$$X(\omega) := \sum_{j=1}^{\infty} X_j(\omega) e_j, \quad \omega \in \mathbb{R}^\infty.$$

Then $X$ is defined on $\mathbb{R}^\infty$ $\gamma$-a.e., and $X$ is an $H$-valued measurable function. Put

$$\mu := \gamma \circ X^{-1}.$$

Then $\mu$ is a finite Borel measure on $H$ and since $\mu_{i_1,\cdots,i_n} = \gamma \circ (X_{i_1}, \cdots, X_{i_n})^{-1}$ we obtain

$$\begin{aligned} \widehat{\mu}\left(\sum_{j=1}^{n} \langle x, e_j\rangle e_j\right) &= f_{1,\cdots,n}\left(\langle x, e_1\rangle, \cdots, \langle x, e_n\rangle\right) \\ &= \phi\left(\sum_{j=1}^{n} \langle x, e_j\rangle e_j\right). \end{aligned}$$

By letting $n \longrightarrow \infty$ we obtain $\widehat{\mu} = \phi$ and the equivalence (1)$\Longleftrightarrow$(2) is proved.
(2) $\Longrightarrow$ (3): Assume that (2) holds. In (2) take $\varepsilon = \frac{1}{k}$ for $k \in \mathbb{N}$ and $\lambda_k > 0$ such that $\sum_{k=1}^{\infty} \lambda_k \mathrm{Tr} Q_{\frac{1}{k}} < \infty$. Set $Q := \sum_{k=1}^{\infty} \lambda_k Q_{\frac{1}{k}}$. It is obvious that $Q$ is a positive symmetric operator of trace class on $H$. Moreover $Q$ satisfies

$$\begin{aligned} \langle Qx, x\rangle < \lambda_k \quad &\Rightarrow \quad \langle Q_{\frac{1}{k}} x, x\rangle < 1 \\ &\Rightarrow \quad \Re\left(\phi(0) - \phi(x)\right) < \frac{1}{k}. \end{aligned}$$

So, by Lemma 1.1.3, $\phi$ is continuous on $H$ with respect to $\|\cdot\|_Q$ and hence (3) is proved.
(3) $\Longrightarrow$ (2): Conversely, suppose (3) is satisfied. So for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$|x|_Q < \delta \Rightarrow \Re\left(\phi(0) - \phi(x)\right) < \varepsilon.$$

Set $Q_\varepsilon := \delta^{-1}Q$. Then,

$$\langle Q_\varepsilon x, x\rangle < 1 \Rightarrow \Re\left(\phi(0) - \phi(x)\right) < \varepsilon$$

and $Q_\varepsilon$ satisfies (2). □

## 1.2 GAUSSIAN MEASURES ON HILBERT SPACES

We will study a special class of Borel probability measures on $H$. We first introduce the notions of **mean vectors** and **covariance operators** for general Borel probability measures.

**Definition 1.2.1** *Let $\mu$ be a Borel probability measure on $H$. If for any $x \in H$ the function $z \mapsto \langle x, z\rangle$ is integrable with respect to $\mu$, and there exists an element $m \in H$ such that*

$$\langle m, x\rangle = \int_H \langle x, z\rangle\, \mu(dz), \quad x \in H,$$

*then $m$ is called the* **mean vector** *of $\mu$. If furthermore there is a positive symmetric linear operator $B$ on $H$ such that*

$$\langle Bx, y\rangle = \int_H \langle z - m, x\rangle\langle z - m, y\rangle\, \mu(dz), \quad x, y \in H,$$

*then $B$ is called the* **covariance operator** *of $\mu$.*

Mean vectors and covariance operators do not necessarily exist in general. But if $\int_H |x|\mu(dx) < \infty$ , then by Riesz' representation theorem, the mean vector $m$ exists, and $|m| \leq \int_H |x|\mu(dx)$. If furthermore, $\int_H |x|^2\mu(dx) < \infty$, then by Lemma 1.1.4, there is a positive symmetric trace class operator $Q$ such that

$$\langle Qx, y\rangle = \int_H \langle x, z\rangle\langle y, z\rangle\, \mu(dz) \quad x, y \in H.$$

Set $Bx = Qx - \langle m, x\rangle m,\ x \in H$. Then it is easy to verify that $B$ is the covariance operator of $\mu$. Note that $B$ is also a positive symmetric trace class operator.

We introduce now Gaussian measures.

**Definition 1.2.2** *Let $\mu$ be a Borel probability measure on $H$. If for any $x \in H$ the random variable $\langle x, \cdot \rangle$ has a Gaussian distribution, then $\mu$ is called a* **Gaussian measure***.*

**Remark 1.2.3** *The scalar function $\langle x, \cdot \rangle$ has a Gaussian distribution means that there exists a real number $m_x$ and a positive number $\sigma_x$ such that*

$$\widehat{\mu}(x) = \int_H e^{i\langle x,z\rangle} \mu(dz) = \exp\left( im_x - \frac{1}{2}\sigma_x^2 \right), \quad x \in H.$$

In the sequel we will characterize Gaussian measures by means of Fourier transform.

**Lemma 1.2.4** *Let $(\alpha_j)_{j\in\mathbb{N}} \subseteq \mathbb{R}$ such that $\sum_{j=1}^{\infty} \alpha_j^2 = \infty$. Then there exists a sequence of real numbers $(\beta_j)$ such that*

$$\alpha_j\beta_j \geq 0 \textit{ for all } j \geq 1, \quad \sum_{j=1}^{\infty} \beta_j^2 < \infty \quad \text{and} \quad \sum_{j=1}^{\infty} \alpha_j\beta_j = \infty.$$

**Proof:** Set $n_0 = 0$ and define $n_k$ inductively as follows

$$n_k := \inf\{l : \sum_{j=n_{k-1}+1}^{l} \alpha_j^2 \geq 1\}, \quad k \geq 1.$$

Then, $n_k \nearrow \infty$. Put

$$\beta_j := \frac{\alpha_j}{k+1} \left( \sum_{j=n_k+1}^{n_{k+1}} \alpha_j^2 \right)^{-\frac{1}{2}}, \quad n_k + 1 \leq j \leq n_{k+1}, \; k = 0, 1, \ldots.$$

Then, for all $j \geq 1$, $\alpha_j\beta_j \geq 0$, and

$$\sum_{j=1}^{\infty} \beta_j^2 = \sum_{k=0}^{\infty} \sum_{j=n_k+1}^{n_{k+1}} \beta_j^2 \sum_{k=0}^{\infty} \frac{1}{(k+1)^2} < \infty,$$

$$\begin{aligned} \sum_{j=1}^{\infty} \alpha_j\beta_j &= \sum_{k=0}^{\infty} \sum_{j=n_k+1}^{n_{k+1}} \alpha_j\beta_j \\ &= \sum_{k=0}^{\infty} \frac{1}{k+1} \left( \sum_{j=n_k+1}^{n_{k+1}} \alpha_j^2 \right)^{\frac{1}{2}} \\ &\geq \sum_{k=0}^{\infty} \frac{1}{k+1} = \infty. \end{aligned}$$

□

The following result gives a characterization of Gaussian measures on separable Hilbert spaces.

**Theorem 1.2.5** *A Borel probability measure $\mu$ on $H$ is a Gaussian measure if and only if its Fourier transform is given by*

$$\widehat{\mu}(x) = \exp\left(i < m, x > -\frac{1}{2} < Bx, x >\right),$$

*where $m \in H$, $B$ is a positive symmetric trace class operator on $H$. In this case, $m$ and $B$ are the mean vector and covariance operator of $\mu$ respectively. Moreover,*

$$\int_H |x|^2 \mu(dx) = \mathrm{Tr} B + |m|^2.$$

**Proof:** Let $\mu$ be a Gaussian measure on $H$.
Claim: $\int_H |x|^2 \mu(dx) < \infty$.
By assumption, for any $x \in H$, $\langle x, \cdot \rangle$ has a Gaussian distribution. Thus there are $m_x \in \mathbb{R}$, and $\sigma_x > 0$ such that

$$\widehat{\mu}(x) = \int_H e^{i<x,z>} \mu(dz) = \exp\left(i m_x - \frac{1}{2}\sigma_x^2\right). \tag{1.4}$$

Let $(e_j)$ be an orthonormal basis of $H$. Since $\int_{\mathbb{R}} (\xi - m)^2 \mathcal{N}(m, \sigma^2)(d\xi) = \sigma^2$ and
$\int_{\mathbb{R}} \xi \mathcal{N}(m, \sigma^2)(d\xi) = m$, we have

$$\begin{aligned} \int_H |x|^2 \mu(dx) &= \sum_{j=1}^{\infty} \int_H \langle x, e_j \rangle^2 \mu(dx) \\ &= \sum_{j=1}^{\infty} \int_{\mathbb{R}} x_j^2 \mu(dx_j) \\ &= \sum_{j=1}^{\infty} (\sigma_{e_j}^2 + m_{e_j}^2). \end{aligned}$$

Let $(\beta_j) \subseteq \mathbb{R}$ such that $\beta_j m_{e_j} \geq 0$ and $\sum_{j=1}^{\infty} \beta_j^2 < \infty$. Set

$$\xi(x) := \sum_{j=1}^{\infty} \beta_j \langle e_j, x \rangle$$

By Schwarz'inequality, the above series converges absolutely and

$$|\xi(x)| \leq \left(\sum_{j=1}^{\infty} \beta_j^2\right)^{\frac{1}{2}} |x|, \quad x \in H.$$

Moreover, $\xi$ is linear. So by Riesz'representation theorem there is $z \in H$ such that $\xi(x) = \langle z, x \rangle$, $x \in H$. By assumption $\xi = \langle z, \cdot \rangle$ is a Gaussian variable with a finite mean, i.e., $\sum_{j=1}^{\infty} \beta_j m_{e_j} < \infty$. Now, by Lemma 1.2.4,

$\sum_{j=1}^{\infty} m_{e_j}^2 < \infty$. Thus, in order to prove $\int_H |x|^2 \mu(dx) < \infty$, it suffices to check $\sum_{j=1}^{\infty} \sigma_j^2 < \infty$.
By Theorem 1.1.5, there is a positive, symmetric, trace class operator $Q$ such that

$$\langle Qx, x\rangle \ < 1 \Rightarrow 1 - \Re\widehat{\mu}(x) < \frac{1}{6}.$$

Hence,

$$1 - \exp\left(-\frac{1}{2}\sigma_x^2\right) \leq 1 - \Re\widehat{\mu}(x) \leq 2\langle Qx, x\rangle + \frac{1}{6}, \ \ \forall\, x \in H. \tag{1.5}$$

Without loss of generality we may assume that the kernel of $Q$ is $\{0\}$.
For $x \in H \setminus \{0\}$, set $y := \frac{1}{\sqrt{3<Qx,x>}} x$. Then

$$\sigma_y^2 = \frac{1}{3\langle Qx, x\rangle}\sigma_x^2, \text{ and } \langle Qy, y\rangle = \frac{1}{3}.$$

Replacing $x$ by $y$ in (1.5), we obtain

$$1 - \exp\left(-\frac{\sigma_x^2}{6\langle Qx, x\rangle}\right) \leq \frac{2}{3} + \frac{1}{6}.$$

This implies that

$$\sigma_x^2 \leq (6\log 6)\langle Qx, x\rangle, \quad x \in H.$$

Thus,

$$\sum_{j=1}^{\infty} \sigma_{e_j}^2 \leq (6\log 6)\mathrm{Tr}Q < \infty.$$

Hence, $\int_H |x|^2 \mu(dx) < \infty$ and the claim is proved. So by the remark following Definition 1.2.1 the mean vector $m$ and the covariance operator $B$ of $\mu$ exist. The above notation gives

$$m_x = \int_H \langle x, z\rangle \mu(dz) = \langle m, x\rangle \text{ and}$$

$$\begin{aligned} \sigma_x^2 &= \int_H \langle x, z\rangle^2 \mu(dz) - m_x^2 \\ &= \int_H [\langle x, z\rangle^2 - \langle m, x\rangle^2]\mu(dz) \\ &= \int_H \langle x, z - m\rangle^2 \mu(dz) = \langle Bx, x\rangle. \end{aligned}$$

From (1.4) we obtain

$$\widehat{\mu}(x) = \exp\left(i\langle m, x\rangle - \frac{1}{2}\langle Bx, x\rangle\right), \quad x \in H.$$

Moreover,

$$\int_H |x|^2 \mu(dx) = \sum_{j=1}^{\infty} (\sigma_{e_j}^2 + m_{e_j}^2) = \mathrm{Tr}B + |m|^2$$

which proves the first implication.
Conversely, let $m \in H$ and $B$ be a positive, symmetric, trace class operator, and consider the positive definite functional

$$\phi(x) = \exp\left( i\langle m, x\rangle - \frac{1}{2}\langle Bx, x\rangle \right), \quad x \in H.$$

Set $Qx := Bx + \langle m, x\rangle m$, $x \in H$. Then $Q$ is a positive, symmetric, trace class operator on $H$. Define $|\cdot|_Q$ on $H$ as follows

$$|x|_Q = |Q^{1/2}x| = \langle Qx, x\rangle^{1/2} = \left(\langle Bx, x\rangle + \langle m, x\rangle^2\right)^{1/2}.$$

Then $\phi(x)$ is continuous at $x = 0$ with respect to $|\cdot|_Q$. So by Theorem 1.1.5, $\phi$ is the Fourier transform of some Borel probability measure $\mu$ on $H$. Clearly for any $x \in H$, $\langle x, \cdot\rangle$ is a Gaussian random variable with mean $\langle m, x\rangle$ and covariance $\langle Bx, x\rangle$ under $\mu$. Thus, $\mu$ is a Gaussian measure. □

A Gaussian measure with mean vector $m$ and covariance operator $B$ will be denoted by $\mathcal{N}(m, B)$. We propose now to compute some Gaussian integrals.

**Proposition 1.2.6** *Let $\mathcal{N}(0, B)$ be a Gaussian measure on $H$. Then there is an orthonormal basis $(e_n)$ of $H$ such that $Be_n = \lambda_n e_n$, $\lambda_n \geq 0$, $n \in \mathbb{N}$. Moreover, for any $\alpha < \alpha_0 := \inf_n \frac{1}{\lambda_n}$, we have*

$$\int_H e^{\frac{\alpha}{2}|x|^2} \mathcal{N}(0, B)(dx) = \left( \prod_{k=1}^{\infty} (1 - \alpha\lambda_k) \right)^{-\frac{1}{2}} = (\det(I - \alpha B))^{-\frac{1}{2}}.$$

**Proof:** The first assertion follows from the fact that $B$ is symmetric and positive. Since $\mathrm{Tr}B = \sum_{k=1}^{\infty} \lambda_k < \infty$, it follows that

$$0 \neq \prod_{k=1}^{\infty} (1 - \alpha\lambda_k)^{-\frac{1}{2}} < \infty \quad \text{for } \alpha < \alpha_0.$$

Furthermore,

$$\begin{aligned} \int_H e^{\frac{\alpha}{2}|<x,e_1>|^2} \mathcal{N}(0, B)(dx) &= \int_{\mathbb{R}} e^{\frac{\alpha}{2}\xi^2} \mathcal{N}(0, \lambda_1)(d\xi) \\ &= \frac{1}{\sqrt{2\pi\lambda_1}} \int_{\mathbb{R}} e^{\frac{\alpha}{2}\xi^2} e^{-\frac{\xi^2}{2\lambda_1}}\, d\xi \\ &= (1 - \alpha\lambda_1)^{-\frac{1}{2}}. \end{aligned}$$

In similar way we have

$$\int_H e^{\frac{\alpha}{2}\sum_{k=1}^n |<x,e_k>|^2} \mathcal{N}(0,B)(dx) = \left(\prod_{k=1}^n (1-\alpha\lambda_k)\right)^{-\frac{1}{2}}$$

and the result follows from the monotone convergence theorem. □

Before proving a more general result we propose first to study the transformation of a Gaussian measure by an affine mapping.

**Lemma 1.2.7** *Let $H$ and $\widetilde{H}$ be two separable Hilbert spaces. Consider the affine transformation $F : H \to \widetilde{H}$ defined by $F(x) = Qx + z$, where $Q \in \mathcal{L}(H, \widetilde{H})$ and $z \in \widetilde{H}$. If we set $\mu_F := \mathcal{N}(m,B) \circ F^{-1}$, the measure defined by $\mu_F(A) = \mathcal{N}(m,B)(F^{-1}(A))$, $A \in \mathcal{B}(\widetilde{H})$, then*

$$\mu_F = \mathcal{N}(Qm + z, QBQ^*).$$

**Proof:** Let compute the Fourier transform of $\mu_F$. From Theorem 1.2.5 we obtain

$$\begin{aligned}
\widehat{\mu_F}(x) &= \int_{\widetilde{H}} e^{i\langle x,\widetilde{y}\rangle} \mu_F(d\widetilde{y}) \\
&= \int_H e^{i\langle x,Qy+z\rangle} \mu(dy) \\
&= e^{i\langle x,z\rangle} \int_H e^{i\langle Q^*x,y\rangle} \mu(dy) \\
&= e^{i\langle x,Qm+z\rangle} e^{-\frac{1}{2}\langle QBQ^*x,x\rangle} \\
&= \mathcal{N}(Qm \widehat{+ z, Q}BQ^*)(x)
\end{aligned}$$

for $x \in H$. So the lemma follows from Theorem 1.2.5. □

From the above lemma follows the following result.

**Proposition 1.2.8** *Let $\alpha_0 := \inf_k \frac{1}{\lambda_k}$. Then, for any $\alpha < \alpha_0$,*

$$\int_H e^{\frac{\alpha}{2}|x|^2} \mathcal{N}(m,B)(dx) = (\det(I-\alpha B))^{-\frac{1}{2}} \exp\left(\frac{\alpha}{2}\langle (I-\alpha B)^{-1}m, m\rangle\right).$$

**Proof:** From Lemma 1.2.7 we have

$$\begin{aligned}
\int_H e^{\frac{\alpha}{2}|x|^2}\mathcal{N}(m,B)(dx) &= \int_H e^{\frac{\alpha}{2}|x+m|^2}\mathcal{N}(0,B)(dx)\\
&= e^{\frac{\alpha}{2}|m|^2}\prod_{k=1}^{\infty}\frac{1}{\sqrt{2\pi\lambda_k}}\int_{\mathbb{R}} e^{\frac{\alpha}{2}\xi^2+\alpha m_k\xi}e^{-\frac{\xi^2}{2\lambda_k}}d\xi\\
&= e^{\frac{\alpha}{2}|m|^2}\prod_{k=1}^{\infty}\frac{1}{\sqrt{2\pi\lambda_k}}\int_{\mathbb{R}} e^{-\left[\frac{1-\alpha\lambda_k}{2\lambda_k}\xi^2-\alpha m_k\xi\right]}d\xi\\
&= e^{\frac{\alpha}{2}|m|^2}\prod_{k=1}^{\infty}\frac{1}{\sqrt{2\pi\lambda_k}}e^{\frac{\lambda_k\alpha^2m_k^2}{2(1-\alpha\lambda_k)}}\int_{\mathbb{R}} e^{-\frac{(1-\alpha\lambda_k)}{2\lambda_k}(\xi-\frac{\lambda_k\alpha m_k}{1-\alpha\lambda_k})^2}d\xi\\
&= \prod_{k=1}^{\infty}\frac{1}{\sqrt{2\pi\lambda_k}}e^{\frac{\alpha}{2}m_k^2}e^{\frac{\lambda_k\alpha^2m_k^2}{2(1-\alpha\lambda_k)}}\left(\int_{\mathbb{R}} e^{-\xi^2}d\xi\right)\left(\frac{2\lambda_k}{1-\alpha\lambda_k}\right)^{\frac{1}{2}}\\
&= \prod_{k=1}^{\infty}(1-\alpha\lambda_k)^{-\frac{1}{2}}e^{\frac{\alpha m_k^2}{2(1-\alpha\lambda_k)}}\\
&= (\det(I-\alpha B))^{-\frac{1}{2}}\,e^{\frac{\alpha}{2}\langle(I-\alpha B)^{-1}m,m\rangle}.
\end{aligned}$$

□

**Example 1.2.9** *Let compute the integrals*

(a)

$$\int_H |x|^{2m}\mathcal{N}(0,B)(dx),\quad m\in\mathbb{N},$$

(b)

$$\int_H |My|^2\,\mathcal{N}(0,B)(dy),\quad \textit{where } M\in\mathcal{L}(H).$$

*(a) For the integral in (a) we consider the function*

$$f(\alpha) := \int_H e^{\frac{\alpha}{2}|x|^2}\mathcal{N}(0,B)(dx) = (\det(I-\alpha B))^{-\frac{1}{2}}\quad \textit{for } \alpha<\alpha_0.$$

*Now, it is easy to see that* $(-\infty,\alpha_0)\ni\alpha\mapsto\det(I-\alpha B)$ *is* $C^\infty$ *and*

$$\frac{d}{d\alpha}\det(I-\alpha B) = \mathrm{Tr}(B(I-\alpha B)^{-1})\det(I-\alpha B),\quad \alpha<\alpha_0.$$

*Furthermore we can differentiate under the integral sign. Hence,*

$$\int_H |x|^{2m}\mathcal{N}(0,B)(dx) = 2^m\frac{d^m}{d\alpha^m}\left(\det(I-\alpha B)\right)^{-\frac{1}{2}}_{|\alpha=0}.$$

*This implies that*

$$\int_H |x|^2 \mathcal{N}(0,B)(dx) = \mathrm{Tr}B$$

*and*

$$\int_H |x|^4 \mathcal{N}(0,B)(dx) = 2\mathrm{Tr}B^2 + (\mathrm{Tr}B)^2.$$

*(b) It follows from Lemma 1.2.7 that*

$$\int_H |My|^2\, \mathcal{N}(0,B)(dy) = \int_H |y|^2\, \mathcal{N}(0,MBM^*)(dy).$$

*So by Theorem 1.2.5 we have*

$$\int_H |My|^2\, \mathcal{N}(0,B)(dy) = \mathrm{Tr}(MBM^*) = \mathrm{Tr}(M^*MB). \tag{1.6}$$

By a same computation as above one has

**Proposition 1.2.10** *For any $\alpha,\ m \in H$, we have*

$$\int_H e^{\langle \alpha, x\rangle} \mathcal{N}(m,B)(dx) = e^{\langle \alpha, m\rangle} e^{\frac{1}{2}\langle B\alpha,\alpha\rangle}.$$

We end this section by proving that the positive definite functional on $H$ defined by $\varphi(x) = e^{-\frac{1}{2}|x|^2}$, $x \in H$, is not the Fourier transform of any Borel measures provided that $\dim H = \infty$.

**Proposition 1.2.11** *Let $Q$ be a positive, symmetric operator on a separable Hilbert space $H$. Then the functional*

$$\phi(x) = \exp\left(-\frac{1}{2} < Qx, x >\right), \quad x \in H,$$

*is the Fourier transform of a probability measure on $H$ if and only if Tr$Q < \infty$.*

**Proof:** Suppose that Tr$Q < \infty$. Then $\phi(0) = 1$ and $\phi$ is $|\cdot|_Q$-continuous positive functional on $H$. So by Theorem 1.1.5 there exists a probability measure $\mu$ such that $\widehat{\mu}(x) = \phi(x),\ x \in H$.
To show the converse, assume that there is a probability measure $\mu$ such that

$$\int_H e^{i<x,y>} \mu(dy) = \exp\left(-\frac{1}{2} < Qx, x >\right).$$

Then by Theorem 1.1.5, for any $\varepsilon \in (0, \frac{1}{3})$, there exists a positive, symmetric operator $Q_\varepsilon$ of trace class such that

$$\begin{aligned} < Q_\varepsilon x, x > < 1 \quad &\Rightarrow \quad \phi(0) - \mathrm{Re}\phi(x) < \varepsilon \\ &\Rightarrow \quad < Qx, x > < 3\varepsilon. \end{aligned}$$

Let now $y_0 \in H$ and $< Q_\varepsilon y_0, y_0 >=: c^2$, with $c > 0$. Let $d > c$ arbitrary. Then
$< Q_\varepsilon \frac{y_0}{d}, \frac{y_0}{d} >= \frac{c^2}{d^2} < 1$.
Hence, $< Q\frac{y_0}{d}, \frac{y_0}{d} > < \varepsilon$, i.e. $< Qy_0, y_0 > < \varepsilon d^2$. Letting $d \to c$, we have $< Qy_0, y_0 >\leq \varepsilon < Q_\varepsilon y_0, y_0 >$. Since $y_0$ is arbitrary, we obtain

$$< Qy, y > \leq \varepsilon < Q_\varepsilon y, y >$$

for all $y \in H$. In particular, for an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of $H$, we obtain

$$\mathrm{Tr}Q = \sum_n < Qe_n, e_n > \leq \varepsilon \sum_n < Q_\varepsilon e_n, e_n >= \varepsilon \mathrm{Tr}Q_\varepsilon < \infty.$$

□

As an immediate consequence we obtain that the functional

$$\phi(x) = \exp\left(-\frac{1}{2}|x|^2\right), \quad x \in H,$$

is not the Fourier transform of any probability measure on $H$ if $\dim H = \infty$.

## 1.3 The Hellinger integral and the Cameron-Martin theorem

The **Cameron-Martin formula** permits us to differentiate under the integral sign with respect to Gaussian measures in infinite dimensional Hilbert spaces. This allows us to obtain some regularity results for parabolic equations with many infinitely variables.
First we need some preparations.
We denote by $\mathcal{L}_1^+(H)$ the space of all positive, symmetric operators of trace class on a separable Hilbert space $H$. Let $B \in \mathcal{L}_1^+(H)$ and consider an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of $H$ and a sequence $(\lambda_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}^+$ such that $Be_n = \lambda_n e_n, \ n \in \mathbb{N}$. Suppose also that $\ker B = \{0\}$.
If we denote by $x_n :=< x, e_n >$, then

$$Bx = \sum_{n=1}^\infty \lambda_n x_n e_n \quad \text{and} \quad B^{\frac{1}{2}}x = \sum_{n=1}^\infty \lambda^{\frac{1}{2}} x_n e_n, \quad x \in H.$$

We set also

$$B_n x := \sum_{k=1}^n \lambda_k x_k e_k \quad \text{and} \quad B_n^{-\frac{1}{2}} x := \sum_{k=1}^n \lambda_k^{-\frac{1}{2}} x_k e_k.$$

Let consider, for $a \in H$ and $n \in \mathbb{N}$, the function

$$g_{a,n}(x) := \langle a, B_n^{-\frac{1}{2}} x\rangle = \sum_{k=1}^n \lambda_k^{-\frac{1}{2}} x_k a_k.$$

If $a \in B^{\frac{1}{2}}(H)$ then one can define the function

$$g_a(x) := \sum_{k=1}^{\infty} \lambda_k^{-\frac{1}{2}} x_k a_k, \quad x \in H.$$

The following proposition shows that it is always possible to define $g_a$ as an $L^2(H,\mu)$-function even if $a \notin B^{\frac{1}{2}}(H)$.

**Proposition 1.3.1** *Let $B \in \mathcal{L}_1^+(H)$ with $\ker B = \{0\}$ and $\mu := \mathcal{N}(0,B)$ its corresponding Gaussian measure on $H$. Then the limit*

$$\lim_{n\to+\infty} g_{a,n} =: g_a$$

*exists in $L^2(H,\mu)$. Moreover,*

$$\int_H |g_a(x)|^2 \mu(dx) = |a|^2$$

*for a given $a \in H$.*

**Proof:** We have

$$\begin{aligned}
\int_H |g_{a,n+p}(x) - g_{a,n}(x)|^2 \mu(dx) &= \int_H \left| \sum_{k=n+1}^{n+p} \lambda_k^{-\frac{1}{2}} x_k a_k \right|^2 \mu(dx) \\
&= \sum_{h,k=n+1}^{n+p} (\lambda_h \lambda_k)^{-\frac{1}{2}} a_h a_k \int_H x_h x_k \mu(dx) \\
&= \sum_{k=n+1}^{n+p} \lambda_k^{-1} a_k^2 \int_H x_k^2 \mu(dx) \\
&= \sum_{k=n+1}^{n+p} a_k^2.
\end{aligned}$$

Hence $(g_{a,n})_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(H,\mu)$. Moreover,

$$\int_H |g_{a,n}|^2 \mu(dx) \sum_{k=1}^{n} \frac{1}{\lambda_k} a_k^2 \int_H x_k^2 \mu(dx) \sum_{k=1}^{n} a_k^2$$

and the theorem is proved by letting $n \to \infty$. □

**Remark 1.3.2** *Suppose that $\ker B = \{0\}$ and take $x \in H$ such that $\langle B^{\frac{1}{2}} a, x\rangle = 0$ for all $a \in H$. Hence, $B^{\frac{1}{2}} x = 0$ and so $Bx = 0$, which implies that $x = 0$. This proves that $B^{\frac{1}{2}}(H)$ is dense in $H$. For the converse, let $x \in H$ with $Bx = 0$. Thus, $B^{\frac{1}{2}} x = 0$ and hence, $\langle B^{\frac{1}{2}} x, y\rangle = \langle x, B^{\frac{1}{2}} y\rangle = 0$ for*

*all $y \in H$. Since $\overline{B^{\frac{1}{2}}(H)} = H$, it follows that $x = 0$.*
*By the same arguments as in the proof of Proposition 1.3.1 one can show that $g_a$ is well defined as an $L^2(H,\mu)$–function and*

$$\|g_a\|_{L^2(H,\mu)} = |a| \quad \text{for } a \in \overline{B^{\frac{1}{2}}(H)}.$$

In the sequel we will use the notation

$$g_a(x) := \langle a, B^{-\frac{1}{2}}x\rangle, \quad x \in H.$$

**Proposition 1.3.3** *Let $B \in \mathcal{L}_1^+(H)$ with $\ker B = \{0\}$ and $\mu := \mathcal{N}(0,B)$ its corresponding Gaussian measure on $H$. Then the limit*

$$\lim_{n\to\infty} e^{g_{a,n}} =: e^{g_a}$$

*exists in $L^2(H,\mu)$ for a given $a \in H$. Moreover, for any $a \in H$,*

$$\int_H e^{\langle a, B^{-\frac{1}{2}}x\rangle} \mathcal{N}(0,B)(dx) = e^{\frac{1}{2}|a|^2}.$$

**Proof:** By applying Proposition 1.2.10 we obtain

$$\begin{aligned}
&\int_H |e^{g_{a,n}} - e^{g_{a,m}}|^2 \mu(dx) \\
= &\int_H \left( e^{2\langle B_n^{-\frac{1}{2}}a,x\rangle} - 2e^{\langle B_n^{-\frac{1}{2}}a,x\rangle + \langle B_m^{-\frac{1}{2}}a,x\rangle} + e^{2\langle B_m^{-\frac{1}{2}}a,x\rangle}\right)\mu(dx) \\
= &\; e^{2\sum_{k=1}^n a_k^2} + e^{2\sum_{k=1}^m a_k^2} - 2\int_H e^{\langle (B_n^{-\frac{1}{2}} + B_m^{-\frac{1}{2}})a,x\rangle}\mu(dx) \\
= &\; e^{2\sum_{k=1}^n a_k^2} + e^{2\sum_{k=1}^m a_k^2} - 2e^{2\sum_{k=1}^n a_k^2 + \frac{1}{2}\sum_{k=n+1}^m a_k^2} \\
= &\; e^{2\sum_{k=1}^n a_k^2}\left(1 + e^{2\sum_{k=n+1}^m a_k^2} - 2e^{\frac{1}{2}\sum_{k=n+1}^m a_k^2}\right) \longrightarrow 0 \;\; (n,m\to\infty).
\end{aligned}$$

This proves that $(e^{g_{a,n}})$ is a Cauchy sequence in $L^2(H,\mu)$ and one can see that

$$\int_H e^{\langle a, B^{-\frac{1}{2}}x\rangle} \mathcal{N}(0,B)(dx) = e^{\frac{1}{2}|a|^2}$$

is satisfied for every $a \in H$. □

We propose now to recall the definition of the **Hellinger** integral.
Let $\mu$, $\nu$ be two probability measures on a measurable space $(\Omega, \Sigma)$. We say that $\mu$ and $\nu$ are **singular** (notation: $\mu \perp \nu$) if there is a set $B \in \Sigma$ such that $\mu(B) = 0$ and $\nu(\Omega \setminus B) = 0$. It is easy to see that two probability measures $\mu$ and $\nu$ are singular if and only if for any $\varepsilon > 0$ there is $B \in \Sigma$ such that $\mu(B) < \varepsilon$ and $\nu(\Omega \setminus B) < \varepsilon$. Further, $\mu$ is called $\nu$-**absolutely continuous** (notation: $\mu \prec \nu$) if $\nu(B) = 0$ implies $\mu(B) = 0$ for any $B \in \Sigma$. So by the theorem of Radon-Nikodym we know that if $\mu$ is $\nu$-absolutely continuous,

then there is a non-negative measurable function $\varphi$ defined on $\Omega$, called the **density function** of $\mu$, such that

$$\mu(B) = \int_B \varphi(\omega)\nu(d\omega)$$

for any $B \in \Sigma$. The density $\varphi$ is denoted by

$$\varphi(\omega) := \frac{d\mu}{d\nu}(\omega), \quad \omega \in \Omega.$$

If $\mu \prec \nu$ and $\nu \prec \mu$ are satisfied then $\mu$ and $\nu$ are called **equivalent** (notation: $\mu \sim \nu$). If $\mu \sim \nu$, then the two density functions $\varphi = \frac{d\mu}{d\nu}$ and $\psi = \frac{d\nu}{d\mu}$ satisfy $\varphi(\omega)\psi(\omega) = 1$, a.e. $\omega \in \Omega$. Hence, $\varphi(\omega) > 0$ $\mu$-a.e. $\omega \in \Omega$.

Let now $\mu$ and $\nu$ two arbitrary probability measures on $(\Omega, \Sigma)$. Let $\gamma$ be a probability measure on $(\Omega, \Sigma)$ such that $\mu \prec \gamma$ and $\nu \prec \gamma$. Such a measure exists, we have to take for example $\gamma = \frac{1}{2}(\mu + \nu)$. Thus, the following integral is well-defined

$$H(\mu, \nu) := \int_\Omega \sqrt{\frac{d\mu}{d\gamma}(\omega)\frac{d\nu}{d\gamma}(\omega)}\,\gamma(d\omega).$$

This integral will be called the **Hellinger** integral.

Let now consider the measurable space $(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty))$, where $\mathcal{B}(\mathbb{R}^\infty)$ is the Borel field of subsets $B$ of $\mathbb{R}^\infty$. On $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ we consider two sequences of measures $(\mu_n)$ and $\nu_n)$ with

$$\mu_n \sim \nu_n, \quad \forall n \in \mathbb{N}. \tag{1.7}$$

Then one has

$$H(\mu_n, \nu_n) = \int_\mathbb{R} \sqrt{\frac{d\nu_n}{d\mu_n}(x_n)}\,\mu_n(dx_n).$$

Let us consider two infinite product measures

$$\mu := \prod_{n=1}^\infty \mu_n \text{ and } \nu := \prod_{n=1}^\infty \nu_n$$

defined on $(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty))$. The following result is du to S. Kakutani [21] and gives a condition under which these two measures $\mu$ and $\nu$ are equivalent.

**Theorem 1.3.4** *Assume that (1.7) is satisfied. Then the following assertions hold.*

(i) *If* $\prod_{n=1}^\infty H(\mu_n, \nu_n) > 0$ *then* $\mu \sim \nu$ *and*

$$\frac{d\nu}{d\mu}(x) = \prod_{k=1}^\infty \frac{d\nu_k}{d\mu_k}(x), \quad \text{a.e. } x \in \mathbb{R}^\infty.$$

(ii) *If* $\prod_{n=1}^{\infty} H(\mu_n,\nu_n) = 0$ *then* $\mu \perp \nu$.

*Moreover,*

$$H(\mu,\nu) = \prod_{n=1}^{\infty} H(\mu_n,\nu_n). \tag{1.8}$$

**Proof:** If we set $\psi_n(x) := \prod_{k=1}^{n} \sqrt{\frac{d\nu_k}{d\mu_k}(x_k)}$ for $x \in \mathbb{R}^\infty$ and $n \in \mathbb{N}$, then

$$\|\psi_n\|^2_{L^2(\mathbb{R}^\infty,\mu)} = \int_{\mathbb{R}^\infty} \prod_{k=1}^{n} \frac{d\nu_k}{d\mu_k}(x_k)\, \mu(dx) = \prod_{k=1}^{n} \int_{\mathbb{R}} \nu_k(dx_k) = 1 \text{ and}$$

$$\begin{aligned}
\|\psi_n - \psi_m\|^2_{L^2(\mathbb{R}^\infty,\mu)} &= \int_{\mathbb{R}^\infty} \left( \prod_{k=1}^{n} \sqrt{\frac{d\nu_k}{d\mu_k}(x_k)} - \prod_{k=1}^{m} \sqrt{\frac{d\nu_k}{d\mu_k}(x_k)} \right)^2 \mu(dx) \\
&= \int_{\mathbb{R}^\infty} \prod_{k=1}^{n} \frac{d\nu_k}{d\mu_k}(x_k) \left( 1 - \prod_{k=n+1}^{m} \sqrt{\frac{d\nu_k}{d\mu_k}(x_k)} \right)^2 \mu(dx) \\
&= 2\left( 1 - \prod_{k=n+1}^{m} \int_{\mathbb{R}} \sqrt{\frac{d\nu_k}{d\mu_k}(x_k)}\, \mu_k(dx_k) \right) \\
&= 2\left( 1 - \prod_{k=n+1}^{m} H(\mu_k,\nu_k) \right) \qquad (1.9)
\end{aligned}$$

for any positive integers $n$ and $m$ with $n < m$.
(i) If $\prod_{n=1}^{\infty} H(\mu_n,\nu_n) > 0$ then

$$\lim_{n,m\to\infty} \prod_{k=n+1}^{m} H(\mu_k,\nu_k) = 1.$$

Hence, by (1.9), $(\psi_n)$ is a Cauchy sequence in $L^2(\mathbb{R}^\infty,\mu)$ and so there is $\psi \in L^2(\mathbb{R}^\infty,\mu)$ such that $\lim_{n\to\infty} \|\psi_n - \psi\|_{L^2(\mathbb{R}^\infty,\mu)} = 0$.
Let prove now that $\nu \prec \mu$ and $\frac{d\nu}{d\mu}(x) = (\psi(x))^2$, $x \in \mathbb{R}^\infty$, i.e.

$$\nu(B) = \int_B (\psi(x))^2\, \mu(dx)$$

for any $B \in \mathcal{B}(\mathbb{R}^\infty)$. To this purpose it follows from Hölder's inequality and (1.9) that

$$\begin{aligned}
&\left(\int_{\mathbb{R}^\infty} |\psi_m(x)^2 - \psi_n(x)^2|\,\mu(dx)\right)^2 \\
\leq\ & \int_{\mathbb{R}^\infty} |\psi_m(x) + \psi_n(x)|^2\,\mu(dx) \int_{\mathbb{R}^\infty} |\psi_m(x) - \psi_n(x)|^2\,\mu(dx) \\
\leq\ & 4\int_{\mathbb{R}^\infty} |\psi_m(x) - \psi_n(x)|^2\,\mu(dx) \\
=\ & 8\left(1 - \prod_{k=n+1}^{m} H(\mu_k, \nu_k)\right)
\end{aligned}$$

for $n < m$. Thus,

$$\lim_{n\to\infty} \|\psi_n^2 - \psi^2\|_{L^1(\mathbb{R}^\infty,\mu)} = 0.$$

Finally let $B \in \mathcal{B}(\mathbb{R}^\infty)$ and set $\chi_n(x) := \chi_B(P_n x)$, $x \in \mathbb{R}^\infty$, where $\chi_B(\cdot)$ denotes the characteristic function of the measurable set $B$ and $P_n x := (x_1, \ldots, x_n, 0, \ldots)$. So we have

$$\begin{aligned}
\int_{\mathbb{R}^\infty} \chi_n(x)\,\nu(dx) &= \int_{\mathbb{R}^n} \chi_B(x_1, \ldots, x_n, 0, \ldots)\,\nu_1(dx_1)\ldots\nu_n(dx_n) \\
&= \int_{\mathbb{R}^n} \chi_n(x) \prod_{k=1}^{n} \frac{d\nu_k}{d\mu_k}(x_k) \prod_{k=1}^{n} \mu_k(dx_k) \\
&= \int_{\mathbb{R}^\infty} \chi_n(x)\psi_n(x)^2\,\mu(dx).
\end{aligned}$$

Since $\psi_n^2 \to \psi^2$ in $L^1(\mathbb{R}^\infty, \mu)$ and by letting $n \to \infty$ we obtain

$$\nu(B) = \int_{\mathbb{R}^\infty} \psi(x)^2\,\mu(dx).$$

In a similar way one can see that $\mu \prec \nu$. So we obtain $\mu \sim \nu$. Finally, since $\mu \sim \nu$, we have

$$\begin{aligned}
H(\mu,\nu) &= \int_{\mathbb{R}^\infty} \psi(x)\,\mu(dx) \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^\infty} \psi_n(x)\,\mu(dx) \\
&= \lim_{n\to\infty} \prod_{k=1}^{n} \int_{\mathbb{R}} \sqrt{\frac{d\nu_k}{d\mu_k}(x_k)}\,\mu_k(dx_k) \\
&= \lim_{n\to\infty} \prod_{k=1}^{n} H(\mu_k, \nu_k).
\end{aligned}$$

So we obtain (1.8).
(ii) If $\prod_{k=1}^\infty H(\mu_k,\nu_k) = 0$ then for any $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that $\prod_{k=1}^n H(\mu_k,\nu_k) < \varepsilon$. Let $B_n \in \mathcal{B}(\mathbb{R}^n)$ with

$$B_n := \{(x_1,\dots,x_n) \in \mathbb{R}^n : \psi_n(x_1,\dots,x_n,0,\dots)^2 = \prod_{k=1}^n \frac{d\nu_k}{d\mu_k}(x_k) > 1\}.$$

Then,

$$\begin{aligned}
\left(\prod_{k=1}^n \mu_k\right)(B_n) &= \int_{B_n} \left(\prod_{k=1}^n \mu_k\right)(dx) \\
&< \int_{B_n} \psi_n(x_1,\dots,x_n,0,\dots)\left(\prod_{k=1}^n \mu_k\right)(dx) \\
&= \int_{B_n} \prod_{k=1}^n \sqrt{\frac{d\nu_k}{d\mu_k}(x_k)}\left(\prod_{k=1}^n \mu_k\right)(dx) \\
&\le \prod_{k=1}^n H(\mu_k,\nu_k) < \varepsilon.
\end{aligned}$$

By the same computation we obtain

$$\left(\prod_{k=1}^n \nu_k\right)(\mathbb{R}^n \setminus B_n) \le \prod_{k=1}^n H(\mu_k,\nu_k) < \varepsilon.$$

Therefore, if we set $B := B_n \times \prod_{k=n+1}^\infty \mathbb{R}$, then

$$\mu(B) < \varepsilon \text{ and } \nu(\mathbb{R}^\infty \setminus B) < \varepsilon.$$

This proves that $\mu\perp\nu$. Suppose now that $\mu\perp\nu$. Then there exists $B \in \mathcal{B}(\mathbb{R}^\infty)$ such that $\mu(B) = 0$ and $\nu(\mathbb{R}^\infty \setminus B) = 0$. So by Hölder's inequality, it follows that

$$\begin{aligned}
H(\mu,\nu) &= \int_B \sqrt{\frac{d\mu}{d\gamma}(x)\frac{d\nu}{d\gamma}(x)}\,\gamma(dx) + \int_{\mathbb{R}^\infty\setminus B} \sqrt{\frac{d\mu}{d\gamma}(x)\frac{d\nu}{d\gamma}(x)}\,\gamma(dx) \\
&\le \left(\int_B \frac{d\mu}{d\gamma}(x)\,\gamma(dx)\right)^{\frac12}\left(\int_B \frac{d\nu}{d\gamma}(x)\,\gamma(dx)\right)^{\frac12} + \\
&\quad \left(\int_{\mathbb{R}^\infty\setminus B} \frac{d\mu}{d\gamma}(x)\,\gamma(dx)\right)^{\frac12}\left(\int_{\mathbb{R}^\infty\setminus B} \frac{d\nu}{d\gamma}(x)\,\gamma(dx)\right)^{\frac12} \\
&= \mu(B)^{\frac12}\nu(B)^{\frac12} + \mu(\mathbb{R}^\infty \setminus B)^{\frac12}\nu(\mathbb{R}^\infty \setminus B)^{\frac12} = 0.
\end{aligned}$$

Therefore, (1.8) holds. This end the proof of the theorem. □

Let prove now the **Cameron-Martin formula**. We note here that the measure space $(H,\mathcal{B}(H))$ can be identified with $(\mathbb{R}^\infty,\mathcal{B}(\mathbb{R}^\infty))$.

**Corollary 1.3.5** *Let $B \in \mathcal{L}_1^+(H)$ such that $\ker B = \{0\}$ and $\mu := \mathcal{N}(0, B)$ and $\nu := \mathcal{N}(m, B)$ be two Gaussian measures on $(H, \mathcal{B}(H))$. Then the following assertions hold.*

(i) *The Gaussian measures $\mu$ and $\nu$ are equivalent if and only if $m \in B^{\frac{1}{2}}(H)$. Moreover the Radon-Nikodym derivative is given by*

$$\frac{d\nu}{d\mu}(x) = \exp\left(-\frac{1}{2}|B^{-\frac{1}{2}}m|^2 + < B^{-\frac{1}{2}}x, B^{-\frac{1}{2}}m >\right).$$

(ii) *The measures $\mu$ and $\nu$ are singular if and only if $m \notin B^{\frac{1}{2}}(H)$.*

**Proof:** We will apply Theorem 1.3.4 to the Gaussian measures $\mu$ and $\nu$. To this purpose let compute the associated Hellinger integral using (1.8). It follows from Proposition 1.2.10 that

$$\begin{aligned} H(\mu_k, \nu_k) &= \int_{\mathbb{R}} \sqrt{\frac{d\nu_k}{d\mu_k}(x_k)}\, \mu_k(dx_k) \\ &= e^{-\frac{m_k^2}{4\lambda_k}} \int_{\mathbb{R}} e^{\frac{m_k x_k}{2\lambda_k}}\, \mathcal{N}(0, \lambda_k)(dx_k) \\ &= e^{-\frac{m_k^2}{8\lambda_k}}. \end{aligned}$$

So by (1.8) we obtain

$$H(\mu, \nu) = \prod_{k=1}^{\infty} e^{-\frac{m_k^2}{8\lambda_k}}.$$

This implies that

$$\begin{aligned} H(\mu, \nu) > 0 \quad &\Longleftrightarrow \quad \sum_{k=1}^{\infty} \frac{m_k^2}{\lambda_k} < \infty \\ &\Longleftrightarrow \quad m \in B^{\frac{1}{2}}(H). \end{aligned}$$

Moreover, in this case, it follows from Theorem 1.3.4 that

$$\begin{aligned} \frac{d\nu}{d\mu}(x) &= \prod_{k=1}^{\infty} \frac{d\nu_k}{d\mu_k}(x) \\ &= \prod_{k=1}^{\infty} e^{-\frac{m_k^2}{2\lambda_k}} e^{\frac{x_k m_k}{\lambda_k}} \\ &= \exp\left(-\frac{1}{2}|B^{-\frac{1}{2}}m|^2 + \langle B^{-\frac{1}{2}}x, B^{-\frac{1}{2}}m\rangle\right), \end{aligned}$$

where $x = \sum_{k=1}^{\infty} x_k e_k$ with $x_k := \langle x, e_k\rangle$ for an orthonormal basis $(e_n)$ of $H$ such that $Be_n = \lambda_n e_n$ for $n \in \mathbb{N}$. Here we used Proposition 1.3.3.
Finally it is clear that the measures $\mu$ and $\nu$ are singular if and only if $m \notin B^{\frac{1}{2}}(H)$. □

**Exercise 1.3.6 (The Feldman-Hajek theorem)**
*Let consider two linear operators* $B_1, B_2 \in \mathcal{L}_1^+(H)$ *with* $\ker B_1 = \ker B_2 = \{0\}$ *and an orthonormal basis* $(e_n)$ *of* $H$ *such that* $B_1 e_n = \lambda_n e_n$, $n \in \mathbb{N}$, *where* $\lambda_n > 0$ *for all* $n \in \mathbb{N}$. *On* $(H, \mathcal{B}(H))$ *we consider the Gaussian measures* $\mu_1 := \mathcal{N}(0, B_1)$ *and* $\mu_2 := \mathcal{N}(0, B_2)$.

1. **The commutative case:** *Suppose that* $B_1 B_2 = B_2 B_1$. *By using Theorem 1.3.4 show that*

   a. *if* $\sum_{n=1}^{\infty} \frac{(\lambda_n - \alpha_n)^2}{(\lambda_n + \alpha_n)^2} < \infty$, *then* $\mu_1 \sim \mu_2$. *In this case*

   $$\frac{d\mu_2}{d\mu_1}(x) = \prod_{n=1}^{\infty} exp\left(-\frac{(\lambda_n - \alpha_n)}{2\lambda_n \alpha_n}\langle x, e_n\rangle^2\right),$$

   b. *if* $\sum_{n=1}^{\infty} \frac{(\lambda_n - \alpha_n)^2}{(\lambda_n + \alpha_n)^2} = \infty$, *then* $\mu_1 \perp \mu_2$.

   *Here* $\alpha_n > 0$, $n \in \mathbb{N}$, *are such that* $B_2 e_n = \alpha_n e_n$, $n \in \mathbb{N}$.

2. **The General case:**

   (a) *Assume that there is* $S \in \mathcal{L}_2^+(H)$ *such that*

   $$B_2 = B_1^{\frac{1}{2}}(Id - S)B_1^{\frac{1}{2}}.$$

   *Show that* $\mu_1 \sim \mu_2$.

   (b) *Assume that* $S \in \mathcal{L}_1^+(H)$ *and* $\|S\| < 1$. *Show that*

   $$\frac{d\mu_2}{d\mu_1}(x) = [\det(I - S)]^{-\frac{1}{2}} \exp(-\frac{1}{2}\langle S(I-S)^{-1}B_1^{\frac{1}{2}}x, B_1^{\frac{1}{2}}x\rangle), x \in H.$$

   *Here* $\mathcal{L}_2^+(H)$ *is the set of positive Hilbert-Schmidt bounded linear operators on* $H$. *That is,* $B \in \mathcal{L}_2^+(H)$ *if and only if* $B \in \mathcal{L}(H)$, $B$ *positive and* $\sum_{n=1}^{\infty} |Be_n|^2 < \infty$.